The X-Ray Transform and Geometric Inverse Problems

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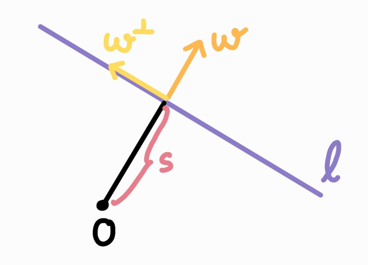
Overview

1. In $\mathbb{R}^2$: the Radon transform
   - Definition
   - What does it have to do with X-Rays?
   - Inversion results
2. General manifolds: the X-Ray transform
   - Introduction and notation
   - Geodesics
   - The Geodesic X-Ray Transform
3. Geometric inverse problems beyond integral transforms

Parametrising the plane

We will use the *parallel-beam geometry* to consider all lines in $\mathbb{R}^2$: each line can be identified with a unit normal vector ω and a distance s from the origin.



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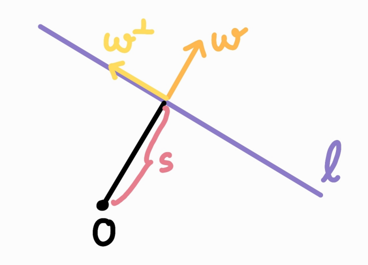

We then have

$$\ell = \{s\omega + t\omega^\perp \mid t \in \mathbb{R}\}$$

where $\omega^\perp$ is the rotation of ω by 90°.

The Radon Transform

Definition

We define the **Radon Transform** of a function $f \in C_c^\infty(\mathbb{R}^2)$ to be

$$Rf(s,\omega) = \int_\ell f = \int_{-\infty}^{\infty} f(s\omega + t\omega^\perp)\,\mathrm{d}t\,, \quad s \in \mathbb{R}, \omega \in S^1$$

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The Radon Transform thus gives a map

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Why do we like the Radon transform?

Because it's closely related to the Fourier transform, which we know loads about!

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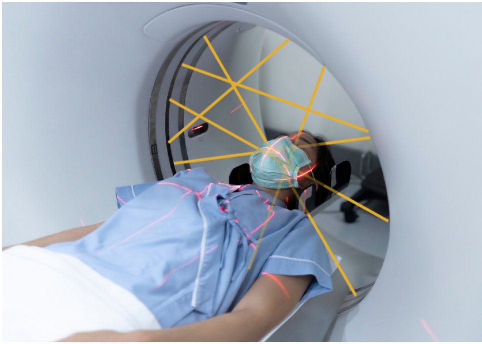

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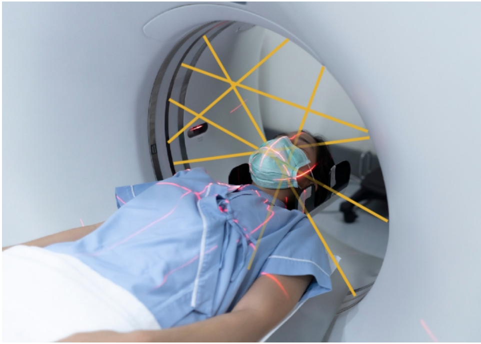

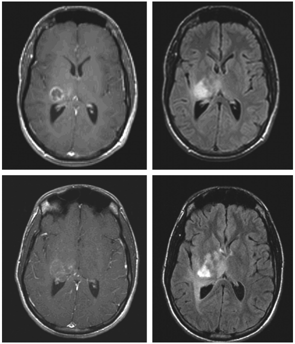

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So inverting Rf gives us the scans we are used to seeing!

Inversion results

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Theorem (Stability)

If $s \in \mathbb{R}$, then for any $f_1, f_2 \in C_c^\infty(\mathbb{R}^2)$, we know that

$$\|f_1 - f_2\|_{H^s(\mathbb{R}^2)} \leq \frac{1}{\sqrt{2}} \|Rf_1 - Rf_2\|_{H_T^{s+1/2}(\mathbb{R}\times S^1)}$$

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So if two Radon transforms are *close*, then the functions themselves must be *close* as well (in appropriate norms).

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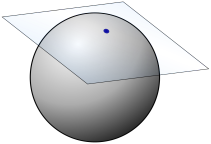

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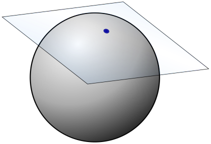

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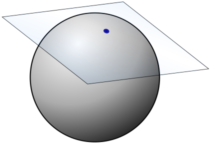

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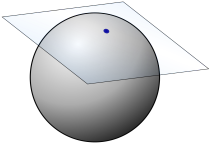

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  The metric also gives a way of measuring lengths, area, volume etc.
- SM - unit sphere bundle (the restriction of the tangent bundle to vectors of unit length)

$$SM = \{(x, v) \in TM \mid \|v\|_g = 1\}$$

Introduction and notation (cont.)

- ∂SM - the boundary of the unit sphere bundle (treated as a manifold)

$$\partial SM = \{(x, v) \in SM \mid x \in \partial M\}$$

This is the same as the restriction of the unit sphere bundle to the boundary of M, but **not** the unit sphere bundle of the boundary $S\partial M$.

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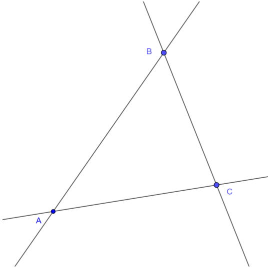
- $\partial_{\pm} SM$ - the two subsets of ∂SM which separate it into vectors pointing *inside* and *outside* the boundary respectively

$$\partial_{\pm} SM = \{(x, v) \in \partial SM \mid \pm\langle v, n(x)\rangle_g \geq 0\}$$

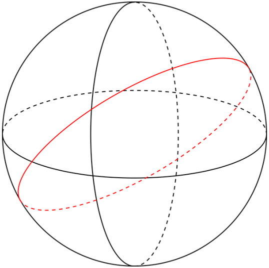
  for $n(x)$ - inward pointing unit normal vector to the boundary.

Geodesics

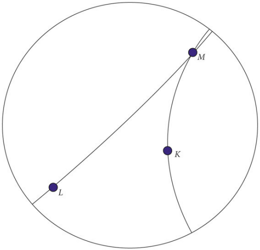
Geodesics on a manifold (M, g) are curves which are *locally* length-minimising.



(a) Euclidean Plane



(b) Sphere



(c) Hyperbolic disc

Existence of geodesics

For each point $(x, v) \in TM$, we know that there is a unique geodesic

$$\gamma_{x,v} : [-\tau_-(x, v), \tau_+(x, v)] \to M$$

such that

$$\gamma_{x,v}(0) = x \qquad \dot{\gamma}_{x,v}(0) = v$$

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Definition

We say M is **non-trapping** if $\tau_+ < \infty$ at all points in TM.

So a non-trapping manifold is one in which all geodesic exit in finite time. We will restrict to such manifolds.

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Assume M is *compact*, *non-trapping*, and has a *smooth boundary*.

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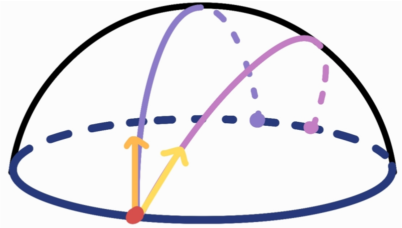
Definition

Given a function $f \in C^\infty(M)$, we define its **geodesic X-Ray transform** to be

$$If(x,v) = \int_0^{\tau_+(x,v)} f(\gamma_{x,v}(t))\,\mathrm{d}t \qquad \forall (x,v) \in \partial_+ SM$$



If we furthermore assume that M has *strictly convex boundary*, then the function If is in $C^\infty(\partial_+ SM)$.
So the Geodesic X-Ray transform gives a linear map

$$I : C^\infty(M) \to C^\infty(\partial_+ SM)$$

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Answer

Not in general, but yes for a large class of manifolds.

The special case

Definition

We say a *compact*, *connected* manifold (M, g) with *smooth boundary* is **simple** if

- It's *non-trapping*
- The boundary is *strictly convex*.
- There are no *conjugate* points.

An intuition for what it means for two points to be *conjugate* is that they can be joined by a one-parameter family of geodesics - *Think of opposite poles on the unit sphere!*

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Equivalent definition

A manifold (M, g) is *simple* if and only if it's compact, connected, has strictly convex boundary, and given any two points there is a unique geodesic connecting them, which depends smoothly on the endpoints.

Injectivity statement

Theorem

If (M, g) is a simple surface, then the Geodesic X-Ray transform I is always injective.

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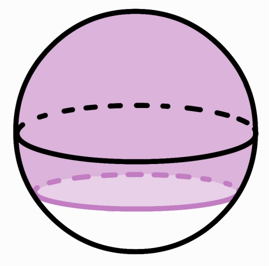

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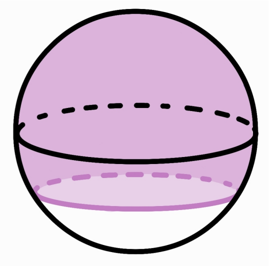

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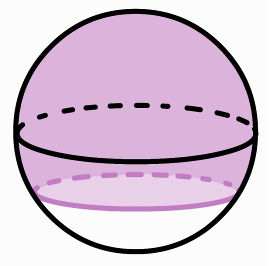

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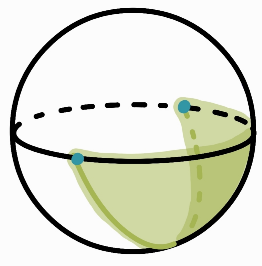

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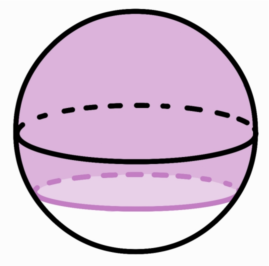

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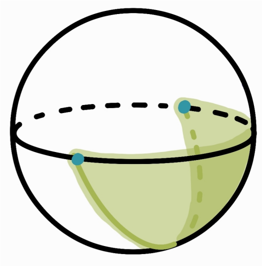

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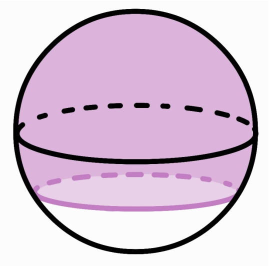

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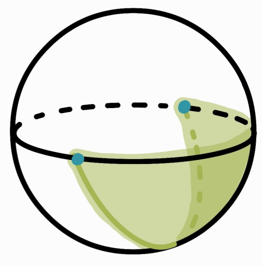

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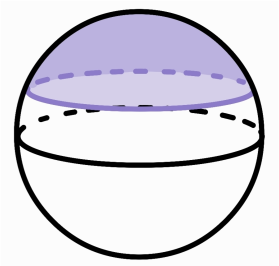

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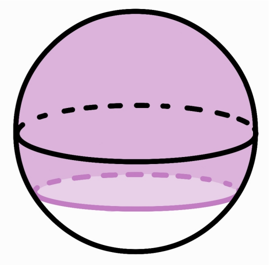

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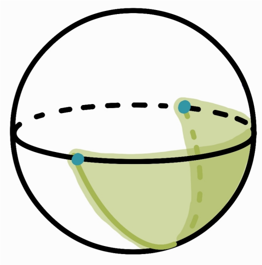

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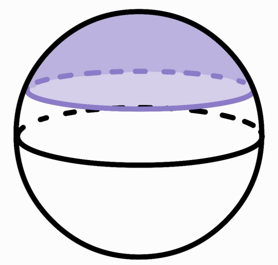

Simple!

Boundary rigidity problem

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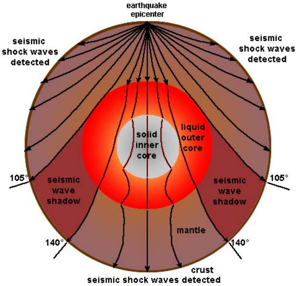
This arose from seismic imaging:
Let the Earth be a ball in $\mathbb{R}^3$, with a metric given by the sound speed in its different substructures (which we'd like to determine). Then earthquakes propagate between points on the Earth's surface by travelling along geodesics, and the time taked in precisely the distance.
So, given lots of measurements of boundary distances, can we invert it to figure out the metric, and hence the composition of the Earth?



Thank you for your attention!

Please do fill in the feedback form if you have the time.