Manifolds with Positive Scalar Curvature

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1 Introduction

In this essay, we will be looking at the problem of prescribing scalar curvature on Riemannian manifolds, that is

Question. *Given a manifold M and a smooth function $f \in C^\infty(M)$, is there a Riemannian metric g on M such that the scalar curvature of (M, g) is equal to f?*

Throughout, all our manifolds will be assumed to be compact, connected, oriented, and without boundary.

The question has a short answer in the case of manifolds of dimension $n \geq 3$, where there are precisely three possibilities for each M, as presented in Aubin's book [Aub98, pages 195-196]:

1. M only admits scalar curvatures which are strictly negative somewhere on M;
2. M only admits scalar curvatures which are either strictly negative somewhere on M or identically zero;
3. M admits any scalar curvature.

In our discussion in this essay, we will only be interested in a weakened version of this, which follows from Kazdan and Warner's work in [KW75]:

Theorem. *Let M be a compact manifold of dimension $n \geq 3$.*

Then M admits any scalar curvature which is strictly negative somewhere on M.

Furthermore, we have three cases:

1. *M admits the metrics above and no others;*
2. *M admits the metrics above and one with identically zero scalar curvature, but no others;*
3. *M admits some metric with positive scalar curvature;*

In fact, the last case is equivalent to M admitting a metric with any scalar curvature, but the proof of this fact involves some considerations of the Yamabe problem which exceed the scope of this essay. The reader is referred to [Aub98] for the details of the stronger result.

However, the theorem as stated above does lead us to the folowing interesting question:

Question. *What topological restrictions are there on a manifold M which admits a metric with positive scalar curvature?*

We will investigate this question using the theory of minimal surfaces, following the work of Schoen-Yau [SY79a], [SY79b].

The case of 3-manifolds will be treated in Section 4, where we will show (Theorem 4.1) that a necessary condition is that the fundamental group of M does not contain any subgroup isomorphic to the fundamental group of a closed surface of positive genus. The approach of that section will be to look at harmonic maps, which will allow us to construct minimal surfaces with certain structures.

Then, in Section 5, we will use an inductive argument to obtain a result for n-manifolds with $4 \leq n \leq 7$. As a corollary of our discussion, we will be able to prove, for $3 \leq n \leq 7$, the following well-known conjecture:

Conjecture (Geroch Conjecture)**.** *The n-torus does not admit a metric with positive scalar curvature.*

2 Two-dimensional manifolds

To warm up, we will start by having a brief discussion of what happens in the case of 2-manifolds.

On a 2-dimensional Riemannian manifold, the scalar curvature R is given by half the Gaussian curvature $\mathcal{K}$ i.e.

$$R = \frac{\mathcal{K}}{2}$$

Thus, we may equivalently study the problem of prescribing the Gaussian curvature.

The key result in this is the following well-known theorem:

Theorem 2.1 (Gauss-Bonnet)**.** *Let M be a two-dimensional (compact, orientable) Riemannian manifold.*

Then

$$\int_M \mathcal{K} = 2\pi\chi(M)$$

(where $\chi(M)$ is the Euler characteristic of M).

As an easy consequence, we can deduce the following:

Corollary 2.2. *Let (M, g) be a two-dimensional compact orientable Riemannian manifold with positive scalar curvature.*

Then M is homeomorphic to a sphere.

Proof. By Gauss-Bonnet, we must have $\chi(M) = \frac{1}{2\pi}\int_M \mathcal{K} > 0$.

But we know that any such surface must be homeomorphic to a sphere, which is exactly what we claimed. □

Thus, there is a very simple topological restriction to which surfaces admit a metric with positive scalar curvature.

3 Rescaling Riemannian metrics and the Negative Scalar Curvature Case

Now, let us establish the trichotomy we claimed earlier:

Theorem 3.1. *Let M be a compact manifold of dimension $n \geq 3$.*

Then M admits any scalar curvature which is strictly negative somewhere on M.

Furthermore, we have three cases:

1. *M admits the metrics above and no others;*
2. *M admits the metrics above and one with identically zero scalar curvature, but no others;*
3. *M admits some metric with positive scalar curvature;*

The proof of this will rely heavily on the idea of conformal rescalings of metrics. This allows us to use an already existing metric to construct new ones with the scalar curvatures we desire.

Much of this section follows the work of Kazdan and Warner [KW75], as well as from Aubin's book [Aub98, Chapter 6].

3.1 Rescaling Riemannian metrics

Let (M, g_0) be a (compact, orientable) Riemannian manifold of dimension $n \geq 3$, and let $R_0 \in C^\infty(M)$ be the corresponding scalar curvature.

For some $f \in C^\infty(M)$, we are interested in the problem of finding a Riemannian metric g on M such that the scalar curvature of (M, g) is given by $R = f$.

We will be looking at the so-called *pointwise conformal rescalings*, i.e. metrics of the form

$$g = u^{\frac{4}{n-2}} g_0$$

for some positive function $u \in C^\infty(M)$.

Under this rescaling, the scalar curvature R_0 transforms as follows:

Proposition 3.2. *If $u > 0$ is a smooth function on M, then the scalar curvature $R \in C^\infty(M)$ of the rescaled metric $g = u^{\frac{4}{n-2}} g_0$ is given by*

$$R = u^{-\frac{n+2}{n-2}} \left(-\frac{4(n-1)}{n-2} \Delta u + R_0 u \right)$$

where Δ is the Laplace-Beltrami operator with respect to the original metric g_0.

The proof of this formula is a know result in the investigation of the Yamabe problem, which can be found, for example, at the start of Yamabe's seminal paper [Yam60].

By the above lemma, the problem of finding a metric with scalar curvature R reduces to finding a positive solution to the partial differential equation

$$-\frac{4(n-1)}{n-2} \Delta u + R_0 u = R u^{\frac{n+2}{n-2}} \tag{1}$$

Absorbing the constants, this PDE is of the form

$$Lu := -\Delta u + hu = H u^a \tag{2}$$

where h, H are smooth functions and $a > 1$.

Remark 3.3. *The existence of a positive u taking scalar curvature R_0 to R is actually an equivalence relation (e.g. if u sends R_0 to R_1 and v sends R_1 to R_2, then uv sends R_0 to R_2).*

Remark 3.4. *It is clear that if $f \in C^\infty(M)$ can be obtained as the scalar curvature R of some metric g, then λf is the scalar curvature of $g_\lambda := \lambda^{-1} g$ for any $\lambda > 0$ (one can check that $u = \lambda^{-\frac{n-2}{4}}$ solves (1)).*

3.2 The first eigenvalue and eigenfunction

We recall from the study of elliptic PDEs that the spectrum of the operator L is given by (possibly finitely many) $\lambda_1 \leq \lambda_2 \leq \ldots$, with $\lambda_k \xrightarrow{k \to \infty} \infty$ if there are infinitely many eigenvalues.

We will focus on studying the first eigenvalue λ_1 of L, together with its eigenspace.

First of all, we have the following result:

Lemma 3.5. *The eigenfunction φ corresponding to the first eigenvalue λ_1 of L can be chosen to be positive.*

A proof of this result in the Euclidean case can be found in [Eva10, Section 6.5.1, Theorem 2], and the case of Riemannian manifolds follows similarly.

In fact, the three cases in Theorem 3.1 actually correspond to the following [Aub98, pg 195-196]:

1. M admits only metrics with $\lambda_1 < 0$.
2. M admits only metrics with $\lambda_1 < 0$ or $\lambda_1 = 0$.
3. M admits metrics with λ_1 of any sign.

Thus, we will be interested in what the metric we started off with can tell us about the value of the first eigenvalue λ_1. One such result is that we have an upper bound for λ_1:

Lemma 3.6. *Denote by $\bar{h}$ the average of h over M i.e.*

$$\bar{h} = \frac{1}{\mathrm{vol}(M)} \int_M h \,\mathrm{d}v$$

Then

$$\lambda_1 \leq \bar{h}$$

Proof. Recall the variational characterisation of the first eigenvalue [Eva10, Section 6.5.1, Theorem 2], which says

$$\lambda_1 = \min_{v \in W^{1,2}(M)} \frac{\langle Lv, v\rangle}{\|v\|_2^2} = \min_{v \in W^{1,2}(M)} \frac{\|\nabla v\|_2^2 + \langle v, hv\rangle}{\|v\|_2^2}$$

Thus, picking $v \equiv 1$ we obtain

$$\lambda_1 \leq \int_M 1 \cdot h = \bar{h}$$

as claimed. □

From this, we deduce the following (incomplete) characterisation of the sign of λ_1:

Lemma 3.7. *Some cases of the sign of λ_1 which can be deduced from properties of h are as follows:*

- *If $\bar{h} < 0$, then $\lambda_1 < 0$.*
- *If $h \equiv 0$, then $\lambda_1 = 0$.*
- *If $h \geq 0$ (pointwise) and not $\equiv 0$, then $\lambda_1 > 0$.*

Proof. The first case is an immediate consequence of Lemma 3.6.

For the second case, the variational characterisation says

$$\lambda_1 = \min_{v \in W^{1,2}(M)} \frac{\|\nabla v\|_2^2 + \langle v, hv\rangle}{\|v\|_2^2} = \min_{v \in W^{1,2}(M)} \frac{\|\nabla v\|_2^2}{\|v\|_2^2} = 0,$$

achieved for $v \equiv 1$.

Finally, if $h \geq 0$ and not identically zero, we have

$$\lambda_1 = \|\nabla \varphi\|_2^2 + \langle \varphi, h\varphi\rangle \geq \int_M h\varphi^2 \,\mathrm{d}v > 0$$

since φ is positive and h is not identically zero. □

3.3 Upper and Lower solutions

The method used by Kazdan and Warner in [KW75] to solve the PDE (2) is that of upper and lower solutions.

Definition 3.1 (Upper and lower solutions)**.** *Consider some $p > n = \dim M$.*

*We say $u_+ \in W^{2,p}(M)$ is an **upper solution** to* (2) *if*

$$Lu_+ \geq Hu_+^a$$

*Similarly, $u_- \in W^{2,p}(M)$ is a **lower solution** if*

$$Lu_- \leq Hu_-^a$$

If we can find lower and upper solutions to (2), we can actually obtain a solution to the PDE which is "sandwiched" between them.

Lemma 3.8. *Suppose $h, H \in C^\infty(M)$ and that we have lower and upper solutions*

$$0 < u_- \leq u_+$$

Then there exists a smooth solution $u \in C^\infty(M)$ to (2) *with $u_- \leq u \leq u_+$.*
In particular, this solution is positive.

We will not prove this lemma here. For some details on the proof, see [KW75, Lemma 2.6].

3.4 The $\lambda_1 < 0$ case

When we know that $\lambda_1 < 0$, it is easy to find lower solutions:

Lemma 3.9. *If $\lambda_1 < 0$, then given any positive continuous function $u \in C(M)$, we can find a lower solution u_- such that $0 < u_- \leq u$.*

Proof. Set $u_- = \alpha\varphi$ for φ the first eigenfunction of L and a constant $\alpha > 0$. Clearly then $u_- > 0$, since both α and φ are positive.

Picking α in such a way that

$$\alpha \leq \left(\frac{|\lambda_1|}{\|H\|_\infty}\right)^{\frac{1}{a-1}} \inf_M \left(\frac{1}{\varphi}\right) \quad \text{and} \quad \alpha\|\varphi\|_\infty \leq \inf_M u$$

we can ensure that u_- satisfies both $u_- \leq u$ and $Lu_- \leq Hu_-^a$, and thus that it is as desired.

(Note that such a choice of α is possible since H and φ are continuous on a compact manifold, and thus achieve their bounds. Hence, their L_∞ norms are finite, and since $\varphi > 0$, its infimum is also positive.) □

Furthermore, if we also know that $H(x) < 0 \; \forall x \in M$, then finding upper solutions is also easy, and we can actually show the following:

Proposition 3.10. *If $H < 0$ and $\lambda_1 < 0$, then we can find a positive solution to* (2)*.*

Proof. Consider $u_+ = K$ for some constant $K > 0$ to be set later.

Then

$$Lu_+ = hK$$

so picking K such that

$$K \geq \left\| \frac{h}{H} \right\|_\infty^{\frac{1}{a-1}}$$

we can ensure that u_+ is an upper solution.

Now use the positive upper solution u_+ to find a lower solution u_- with $0 < u_- \leq u_+$ using Lemma 3.9, then conclude that (2) has a positive solution by Lemma 3.8. □

In particular, this means that for any h such that $\lambda_1 < 0$, we may always find a rescaling taking it to $\widetilde{h} \equiv -1$. Hence, the problem of solving (2) in this case reduces to finding a solution to the PDE

$$-\Delta u - u = Hu^a$$

We first solve this close (in L^p norm) to the function which is identically equal to -1:

Lemma 3.11. *Let $p > \dim M$. There exists some $\varepsilon > 0$ such that given any $H \in L^p(M)$ with $\|H + 1\|_p < \varepsilon$ we can find a positive solution $u \in W^{2,p}(M)$ to the PDE*

$$-\Delta u - u = Hu^a$$

Furthermore, this solution is smooth on any open set where H is smooth.

Proof. We will follow the proof as given in [Aub98, Theorem 6.2]. The main idea is to apply the Implicit Function Theorem to the map sending a pair (u, H) to the expression we want to be equal to 0. This will tell us that we can write u solving the PDE as a function of H in a sufficiently small neighbourhood of -1, and thus in particular that such a solution must exist.

Consider $\Omega = \left\{ u \in W^{2,p}(M) \;\middle|\; u > 0 \right\}$ (which holds pointwise, and not just almost everywhere, by Sobolev embedding).

The operator $L = -\Delta - 1$ induces a map

$$\begin{aligned} \Gamma : \Omega \times L^p(M) &\to L^p(M) \\ (u, H) &\mapsto Lu - Hu^a = -\Delta u - u - Hu^a \end{aligned}$$

(where $Hu^a \in L^p(M)$ since H is and u is bounded because it is in particular C^1).

From the definition of Γ and the fact that $a > 1$, we can see that Γ is actually continuously differentiable, with partial u-derivative

$$D_u\Gamma(u, H)(v) = -\Delta v - v - aHu^{a-1}v$$

Consider $u_0 \equiv 1$ and $H_0 \equiv -1$. Then

$$\Gamma(u_0, H_0) = -1 - (-1) \cdot 1 = 0$$

and

$$D_u\Gamma(u_0, H_0)(v) = -\Delta v + (a - 1)v$$

and so $D_u\Gamma(u_0, H_0) : W^{2,p}(M) \to L^p(M)$ is invertible (by standard elliptic theory, using $a > 1$).

Thus, the Implicit Function Theorem (Banach space version, see [Aub98, Theorem 3.10]) tells us that there exists some L^p-neighbourhood of $H_0 = -1$ where the equation $\Gamma(u, H) = 0$ has has a solution $u = u(H)$ in some $W^{2,p}$-neighbourhood of $u_0 = 1$ for all H's in this neighbourhood.

WLOG shrinking the neighbourhood, we may assume that the entire neighbourhood is contained in Ω i.e. all u's are positive.

Thus, we have obtained precisely that there exists some $\varepsilon > 0$ such that for any $H \in L^p(M)$ with $\|H + 1\|_p < \varepsilon$, there exists a positive solution $u \in W^{2,p}(M)$ to the PDE

$$-\Delta u - u = Hu^a$$

Finally, the smoothness of the solution follows by a standard bootstrap argument. $\square$

3.5 Somewhere Negative Scalar Curvature

We can now prove the first part of the trichotomy we are interested in, namely:

Proposition 3.12. *Let M be a compact, orientable manifold, and let $f \in C^\infty(M)$ be a function such that $f(x) < 0$ for some $x \in M$.*

Then there exists a Riemannian metric g on M such that the scalar curvature of (M, g) is equal to f.

Proof. Firstly, Elíasson [Elí71, Theorem 3] showed that any compact orientable manifold M of dimension $n \geq 3$ admits a Riemannian metric g with negative total scalar curvature i.e. with

$$\int_M R(x)\,\mathrm{d}v < 0$$

By Lemma 3.7, we know that the total scalar curvature being negative implies that $\lambda_1 < 0$.

Hence, by our previous discussion, we know that M admits a metric g_0 with constant negative scalar curvature -1. We will now deform this into one with the scalar curvature we desire.

Since f is negative somewhere, we can WLOG rescale f to assume that $f(x) = -1$ for some $x \in M$ (by Remark 3.4, this does not change whether M admits a metric with this scalar curvature).

We will make use of the following lemma, which we will prove after we finish proving the current theorem:

Lemma 3.13. *Let (M, g) be a compact orientable Riemannian manifold and $f \in C^\infty(M)$ be a function such that $f(x) = -1$ for some $x \in M$.*

Then, given any $\varepsilon > 0$ and any $p \in [1, \infty)$, there exists some diffeomorphism $\varphi : M \to M$ such that

$$\|f \circ \varphi + 1\|_{L^p} < \varepsilon$$

With ε as in Lemma 3.11, let $\varphi : M \to M$ be the diffeomorphism given by Lemma 3.13.

Then Lemma 3.11 tells us that we can find some metric g' on M with scalar curvature given by $f \circ \varphi$.

But then pulling g' back under φ, we get a metric $g := \varphi^* g'$ on M with scalar curvature given by

$$R_g = R_{g'} \circ \varphi^{-1} = f \circ \varphi \circ \varphi^{-1} = f$$

as required. $\square$

Proof of Lemma 3.13. The idea of the proof is to define a new volume form $\mathrm{d}v'$ on M which concentrates most of the volume on the set where f is close to -1, and then use Moser's trick to find a diffeomorphism relating the two volume forms.

Let $0 < \delta_1, \delta_2 < 1$ be constants to be set later.

Consider the set

$$U = f^{-1}(B_{\delta_1}(-1)) = \{\, x \in M \mid |f(x) + 1| < \delta_1 \,\}$$

and let $V \subset U$ be an open subset of U with $\mathrm{vol}(V) > \frac{1}{2}\,\mathrm{vol}(U)$ and $\overline{V} \subset U$.

We can then consider a smooth function $\chi \in C^\infty(M)$ such that

$$\chi(x) = \begin{cases} 1 & x \in V \\ \delta_2 & x \in M \setminus U \end{cases}$$

and $\chi(x) \in [\delta_2, 1]$ for all $x \in M$.

Denote $\mathrm{d}v$ to be the volume form of the metric g on M, and let $\mathrm{d}v' = C\chi\,\mathrm{d}v$, for a constant $C > 0$ chosen such that

$$\int_M \mathrm{d}v' = \mathrm{vol}(M) = \int_M \mathrm{d}v$$

In particular, we note that

$$\int_M \mathrm{d}v' = C\int_M \chi\,\mathrm{d}v \geq C\,\mathrm{vol}(V) \geq \frac{1}{2}C\,\mathrm{vol}(U)$$

and so

$$C \leq \frac{2\,\mathrm{vol}(M)}{\mathrm{vol}(U)}$$

By Moser, we know that there exists a diffeomorphism $\varphi : M \to M$ such that $\varphi^*\,\mathrm{d}v' = \mathrm{d}v$. Under this diffeomorphism, we have

$$\begin{aligned}
\|f\circ\varphi + 1\|_{L^p}^p &= \int_M |f\circ\varphi + 1|^p\,\mathrm{d}v \\
&= \int_M |f+1|^p\,\mathrm{d}v' \\
&= \int_M |f+1|^p C\chi\,\mathrm{d}v \\
&= C\left(\int_U |f+1|^p\chi\,\mathrm{d}v + \int_{M\setminus U} |f+1|^p\chi\,\mathrm{d}v\right) \\
&\leq C\left(\int_U \delta_1^p\,\mathrm{d}v + \int_{M\setminus U} |f+1|^p\delta_2\,\mathrm{d}v\right) \\
&\leq \frac{2\,\mathrm{vol}(M)}{\mathrm{vol}(U)}\left(\delta_1^p\,\mathrm{vol}(U) + \|f+1\|_\infty^p\delta_2\,\mathrm{vol}(M\setminus U)\right) \\
&\leq 2\delta_1^p\,\mathrm{vol}(M) + 2\|f+1\|_\infty^p\frac{\mathrm{vol}(M)^2}{\mathrm{vol}(U)}\delta_2
\end{aligned}$$

so we can indeed choose δ_1 and then $\delta_2 = \delta_2(U) = \delta_2(\delta_1)$ sufficiently small so that $\|f\circ\varphi + 1\|_{L^p} < \varepsilon$. $\square$

3.6 The $\lambda_1 > 0$ case

For $\lambda_1 > 0$, we no longer have statements which work for a general class of functions H. However, we do have the following result:

Lemma 3.14. *Suppose $\lambda_1 > 0$. Then there exists some everywhere positive $H \in C^\infty(M)$ such that the equation*

$$Lu = Hu^a$$

has a positive solution $u \in C^\infty(M)$.

This tells us that, as soon as we have found a metric with $\lambda_1 > 0$, we know that *some* metric with everywhere positive scalar curvature exists.

Proof. Define $H = \lambda_1 \varphi^{1-a}$ for φ the first eigenfunction of L. Clearly H is everywhere positive.

Then

$$H\varphi^a = \lambda_1 \varphi = L\varphi$$

and so $u = \varphi$ is the positive solution we seek. □

3.7 Non-negative scalar curvature

All that remains now in the proof of Theorem 3.1 is to analyse metrics with non-negative scalar curvature. We want to prove the following:

Theorem 3.15. *Let M be a compact orientable manifold of dimension $n \geq 3$.*

Suppose that there exists a Riemannian metric g_0 on M such that the scalar curvature R_0 of (M, g_0) is non-negative everywhere on M, and not identically equal to 0.

Then there exists a Riemannian metric g on M such that the scalar curvature R of (M, g) is positive everywhere on M.

Proof. By Lemma 3.7, we know that $\lambda_1 > 0$.

Then Lemma 3.14 tells us that M admits a metric with positive scalar curvature. □

4 Positive Scalar Curvature in 3-manifolds

We will now look at 3-manifolds with positive scalar curvature. This was treated by Schoen and Yau in [SY79a], which we will be following closely.

We will introduce the following abbreviations:

Definition 4.1. *We say a Riemannian manifold (M, g) **has PSC** if the scalar curvature R of (M, g) is positive everywhere on M.*

*We say that a manifold M **is PSC** if there exists some Riemannian metric g on M such that (M, g) has PSC. Conversely, we will say that M **is not PSC** if there is no metric g on M such that (M, g) has PSC.*

The main result of this section is the following:

Theorem 4.1. *Let M be a 3-dimensional compact connected orientable manifold.*

Suppose that $\pi_1(M)$ contains a subgroup which is isomorphic to the fundamental group $\pi_1(\Sigma_g)$ of the closed surface of genus $g \geq 1$.

Then M is not PSC.

We will spend most of this section discussing the proof of this statement. Before we do so, let us first illustrate an immediate corollary:

Corollary 4.2 (Geroch conjecture, $n = 3$)**.** *The* 3*-dimensional torus* $T^3 = S^1 \times S^1 \times S^1$ *is not PSC.*

Proof. We know that $\pi_1(\Sigma_1) = \pi_1(T^2) \cong \mathbb{Z}^2$ and $\pi_1(T^3) \cong \mathbb{Z}^3$ (see for example [Hat02, Example 1.13]).
But then clearly $\mathbb{Z}^2 \lesssim \mathbb{Z}^3$, and so Theorem 4.1 tells us that T^3 is not PSC. $\square$

A brief overview of the proof of Theorem 4.1 is as follows:

1. Use the subgroup of $\pi_1(M)$ to obtain a smooth map $\phi : \Sigma_g \to M$ which is injective on fundamental groups.
2. For a given metric on Σ_g, consider the variational problem of minimising the energy of maps $\Sigma_g \to M$ with the same action on fundamental groups as ϕ, and show that a smooth minimiser exists.
3. Use the existence of a smooth minimiser for every metric to look at the problem on the moduli space of all such surfaces (with different metrics). From this, obtain the existence of a minimal immersion of Σ_g into M, and thus a stable minimal smooth hypersurface of positive genus.
4. Show that 3-manifolds with positive scalar curvature do not contain any stable minimal hypersurfaces of positive genus, and hence conclude that M is not PSC.

4.1 Defining the map $\phi : \Sigma_g \to M$

This section is dedicated to proving the following lemma:

Lemma 4.3. *Let M be a compact* 3*-dimensional manifold such that $\pi_1(M)$ contains a subgroup isomorphic to $\pi_1(\Sigma_g)$ for some $g \geq 1$.*
Then there exists a smooth map $\phi : \Sigma_g \to M$ such that $\phi_* : \pi_1(\Sigma_g) \to \pi_1(M)$ is injective.

Proof. Let $G \leq \pi_1(M, *)$ be the subgroup isomorphic to $\pi_1(\Sigma_g, *)$.
Recall that this is the group generated by $2g$ elements, call them $\alpha_1, \beta_1, \dots, \alpha_g, \beta_g$, under the relation

$$[\alpha_1, \beta_1] \cdot \dots \cdot [\alpha_g, \beta_g] = \mathrm{id}$$

Thus, identifying each α_i and β_i with a loop in M, we know that the concatenated loop

$$\alpha_1\beta_1\alpha_1^{-1}\beta_1^{-1} \dots \alpha_g\beta_g\alpha_g^{-1}\beta_g^{-1}$$

is contractible in M. Let the homotopy be given by H.
Now recall that Σ_g can be defined using the $4g$-gon in Figure 1.
We can then define a map $\phi : \Sigma_g \to M$ by sending each edge of the $4g$-gon to the corresponding loop in M, the centre of the $4g$-gon to the basepoint $*$ in M, and interpolating between these through the homotopy H.
Clearly by construction ϕ_* is injective, with $\phi_*(\pi_1(\Sigma_g)) = G$ (more precisely, $\phi_*(a_i) = \alpha_i$ etc)
A priori, the function ϕ we have defined is merely continuous. However, this can always be uniformly approximated by a smooth map (by Whitney Approximation, see for example [Lee13, Theorem 10.21]). Picking the smooth map to be sufficiently close in uniform norm, it will be forced to have same action on fundamental groups.
Thus, we obtain the smooth ϕ whose existence we claimed. $\square$

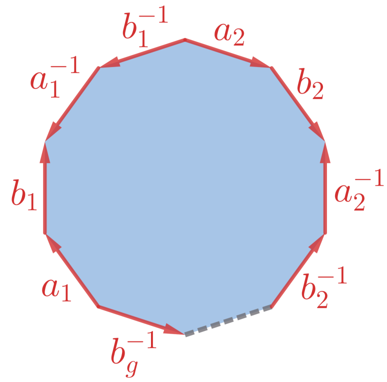

Figure 1: The gluing of the $4g$-gon to obtain Σ_g.

4.2 The variational problem

We now want to "minimise" the area of the map $\phi : \Sigma_g \to M$ which we obtained in the previous section, while preserving the action on fundamental groups.

Thus, we can consider the problem of minimising the energy functional over maps of the form

$$\{ f : \Sigma_g \to M \mid f_* = \phi_* \text{ up to conjugation} \}$$

Since taking minimisers might not always preserve regularity, we will need to consider maps f which are not necessarily smooth. However, this will come at the cost of having to understand what we mean by the induced map on fundamental groups (since now f need not map continuous loops to continuous loops).

We will now make this all precise in the slightly more general setting of $\phi : N \to M$ being any smooth map between compact, oriented Riemannian manifolds. We will assume that N is 2-dimensional.

4.2.1 Function spaces and the energy functional

By Nash Embedding Theorem (see e.g. [Lee13, Theorem 4.34]), we know that there exists an isometric embedding $(M, g) \hookrightarrow (\mathbb{R}^k, g_{\mathrm{Eucl}})$ for some k. Thus, without loss of generality, we may assume that M is a Riemannian submanifold of $\mathbb{R}^k$ (with g corresponding to the standard Euclidean dot product).

Let $\mathrm{d}v$ be the (Riemannian) volume element on N.

We can define the following Sobolev space:

Definition 4.2 ($H^1(N, \mathbb{R}^k)$). *The space $H^1(N, \mathbb{R}^k)$ consists of all functions $f : N \to \mathbb{R}^k$ which are square integrable and have square integrable first derivatives (in the distribution sense), with inner product*

$$(f_1, f_2)_1 = \int_N \langle f_1(x), f_2(x) \rangle \, \mathrm{d}v + \int_N \langle \mathrm{d}f_{1x}, \mathrm{d}f_{2x} \rangle \, \mathrm{d}v$$

Here $\langle \mathrm{d}f_x, \mathrm{d}g_x \rangle$ denotes the inner product as elements of $\mathrm{Hom}(T_xN, \mathbb{R}^k) = T_x^*N \otimes \mathbb{R}^k$ (which inherits its inner product from the one defined by g on T_xN).

This is a subset of the standard $L^2(N, \mathbb{R}^k)$ space. We will denote the L^2 norm and inner product by $\|\cdot\|$ and $(\cdot, \cdot)$ respectively, and the H^1 norm by $\|\cdot\|_1$.

In this space, we can define

Definition 4.3 (energy). *The **energy** of a function $f \in H^1(N, \mathbb{R}^k)$ is defined as*

$$E(f) = \|f\|_1^2 - \|f\|^2 = \int_N \langle \mathrm{d}f, \mathrm{d}f \rangle \, \mathrm{d}v$$

This energy is precisely the quantiy which we will aim to minimise.

A standard compactness and lower semicontinuity argument, which can be found for example in [Mor66, Theorems 1.8.1, 3.4.4] tells us:

Lemma 4.4. *Let $\{f_i\}$ be any bounded sequence in $H^1(N, \mathbb{R}^k)$.*

Then there exists a subsequence $\{f_{i_j}\}$ and a map $f \in H^1(N, \mathbb{R}^k)$ such that f_{i_j} converges to f:

1. *weakly in $H^1(N, \mathbb{R}^k)$,*
2. *pointwise almost everywhere, and*
3. *strongly in $L^2(N, \mathbb{R}^k)$.*

Moreover, we have

$$E(f) \leq \lim_{j \to \infty} E(f_{i_j})$$

Finally, we specialise to maps $N \to M$ by considering the following subspace of $H^1(N, \mathbb{R}^k)$:

Definition 4.4 ($H^1(N, M)$)**.**

$$H^1(N, M) := \{f \in H^1(N, \mathbb{R}^k) \mid f(x) \in M \text{ for almost all } x \in N\}$$

Remark 4.5. *It is clear that this is a closed subset of $H^1(N, \mathbb{R}^k)$ under pointwise convergence, and that Lemma 4.4 also holds if we replace all instances of $\mathbb{R}^k$ by M.*

4.2.2 Action of $H^1(N, M)$ maps on fundamental groups

Consider now a map $f \in H^1(N, M)$.

When f is continuous, we have a well-defined map on fundamental groups $\pi_1(N) \to \pi_1(M)$ given by sending a loop γ in N to the loop $f \circ \gamma$, its image under f. However, this definition does not immediately extend to the less regular case of Sobolev maps, since $f \circ \gamma$ need not be continuous. We will now show that there is still a way to define an induced map $\pi_1(N) \to \pi_1(M)$ up to conjugation (i.e. change of basepoint).

In order to define the action, we instead consider a small tubular neighbourhood of a loop γ. This gives us a 1-parameter of homotopic loops γ^s. The regularity of f will tell us that f is actually continuous along almost all γ^s's, and thus we have a well defined notion of $f_*([\gamma^s]) \in \pi_1(M)$. Finally, we check that each choice of s leads to an equivalent element of $\pi_1(M)$.

Fix some initial basepoint $x_0 \in N$, and consider a set of generators $\{\gamma_i\}_{i=1}^l$ of $\pi_1(N, x_0)$, each a map $\gamma_i : S^1 \cong [-2, 2]/\sim \to N$ (where $-2 \sim 2$) with $\gamma_i(0) = x_0$.

Around each loop, we can pick a tubular neighbourhood i.e. an open set $\gamma_i \subset T_i \subset N$ and a smooth immersion

$$\begin{aligned} \psi_i : S^1 \times [-1, 1] &\to T_i \subset N \\ (\cdot, 0) &\mapsto \gamma_i(\cdot) \end{aligned}$$

From this, we can define a one-parameter family of loops

$$\gamma_i^s = \psi_i(\cdot, s)$$

It is clear by definition that all the loops γ_i^s for a fixed i are (freely) homotopic to each other, through the obvious homotopy arising in $S^1 \times [-1, 1]$.

Without loss of generality, this construction can be made in such a way that actually

$$\gamma_i^s\mid_{(-1,1)} \equiv \gamma_j^s\mid_{(-1,1)} \quad \forall i,j,s$$

In particular, this means that, for fixed s, the loops γ_i^s are all based at the same point, call it $*_s$. Thus, we have a one-parameter family of generators for each $\pi_1(N,*_s)$, given by $\{\gamma_i^s\}$.

Proposition 4.6. *Let $f \in H^1(N,M)$ be a map. Then for almost all $s \in [-1,1]$ the map $f\circ\gamma_i^s : S^1 \to M$ is continuous for all i.*

Thus, we obtain a map

$$f_*^s : \pi_1(N,*_s) \to \pi_1(M, f(*_s))$$

for almost all $s \in [-1,1]$.

Proof. Consider the map $f \circ \psi_i : S^1 \times [-1,1] \to M$.

Then

$$\begin{aligned}\|f\mid_{T_i}\|_1^2 &= \int_{T_i} |f|^2 + |\mathrm{d}f|^2 \\ &\approx \int_{-1}^{1} \left(\int_{S^1} |f\circ\gamma_i^s(t)|^2 + |\mathrm{d}f\circ\gamma_i^s\,(t)|^2\,\mathrm{d}t \right) \mathrm{d}s\end{aligned}$$

where by $\approx$ we mean that the two terms are bounded by constant multiples of each other.

Now, since $\|f\|_1 < \infty$, we must have that

$$\int_{S^1} |f\circ\gamma_i^s|^2 + |\mathrm{d}f\circ\gamma_i^s|^2\,\mathrm{d}t < \infty \quad \text{for almost all } s \in [-1,1]$$

i.e. $f\circ\gamma_i^s \in H^1(S^1,N)$.

Identifying S^1 with the interval $[-2,2]$, an argument which can be found for example in [Bre11, Theorem 8.2] lets us conclude that for all s with $f\circ\gamma_i^s \in H^1(S^1,N)$, the map $f\circ\gamma_i^s$ is (absolutely) continuous.

Finally, since there are finitely many i's, we can conclude that for almost all $s \in [-1,1]$, the map $f\circ\gamma_i^s$ is continuous for all i, as claimed. □

Finally, we must show that these maps are independent of the choice of s. For this, we require the following lemma:

Lemma 4.7. *Let $D \subset \mathbb{R}^2$ be the unit disc.*

There exists some $\varepsilon > 0$ such that for any $f \in H^1(D,M)$ with $E(f) < \varepsilon$, we have that $f\circ\gamma$ is contractible in M whenever γ is a closed simple curve in D and f is absolutely continuous along γ.

Proof. Since M is a compact manifold immersed in $\mathbb{R}^k$, there is some $\delta > 0$ such that the tubular neighbourhood

$$T = \left\{ x \in \mathbb{R}^k \;\middle|\; d(x,M) < \delta \right\}$$

is homotopy equivalent to M. Thus, it suffices to show that $f\circ\gamma$ is contractible in T.

We may approximate the continuous curve $f\circ\gamma \subset M$ by a smooth curve Γ in T which is simple and has $d(x,M) \le \frac{\delta}{2}$ for all $x \in \Gamma$, and with Γ homotopic to $f\circ\gamma$ in T[1]. Thus, it suffices to show that Γ is contractible in T.

[1] This is possible since we're working in $k \ge 4$, and we have enough codimension to *separate* any self-intersection. If $k = 2$, one can easily construct examples where this is not necessarily the case.

Since Γ is a smooth Jordan curve, The Plateau Problem [Mor48, Theorem 3.1] tells us that we can find a minimal embedding $u : D \hookrightarrow M$ with Γ as its boundary. Call Σ the image of this embedding.

Now Σ is homeomorphic to a disc and has boundary Γ, so showing $\Sigma \subset T$ will immediately imply that Γ is contractible in T and thus conclude the proof.

Suppose this is not the case. Consider $x_0 \in \Sigma \setminus T$. Since we assumed that all points on Γ are at most $\frac{\delta}{2}$ away from M, whereas T includes all points which are distance δ away, we know that

$$\Gamma \cap B_{\delta/2}(x_0) = \emptyset$$

Thus, we can apply the Corollary of Theorem 1 in Alexander-Osserman [AO75] to get an estimate on the area of Σ:

$$\text{Area}(\Sigma) \geq \text{Area}(\Sigma \cap B_{\delta/2}(x_0)) \geq \frac{\pi\delta^2}{4}$$

On the other hand, it is known that

$$\text{Area}(\Sigma) \leq E(u)$$

and since u is a minimiser of the Dirichlet energy, picking Γ to be sufficiently close to $f \circ \gamma$ we can ensure that

$$E(u) \leq 2E(f) < 2\varepsilon$$

Setting $\delta > 0$ such that $\frac{\pi\delta^2}{4} > 2\varepsilon$ we get a contradiction. Thus $\Sigma \subset T$ as claimed. $\square$

As a shorthand, we define the following notation:

Definition 4.5 ($\sim_c$). *Let θ_1, θ_2 be two homomorphism $\pi_1(N, x_i) \to \pi_1(M, y_i)$ respectively. We say that $\theta_1 \sim_c \theta_2$ if their action is conjugate i.e. there exist curves γ_1 on N and γ_2 on M, with endpoints $x_1, x_2 \in N$ and $y_1, y_2 \in M$ respectively, such that*

$$\theta_2 = \gamma_2 \circ \theta \left(\gamma_1^{-1} \circ \cdot \circ \gamma_1\right) \circ \gamma_2^{-1}$$

We denote $[\theta]_c$ to be the equivalence class of θ under this relation.

Basically what we do in this definition is declare maps f_1 and f_2 acting similarly but at different basepoints as "the same".

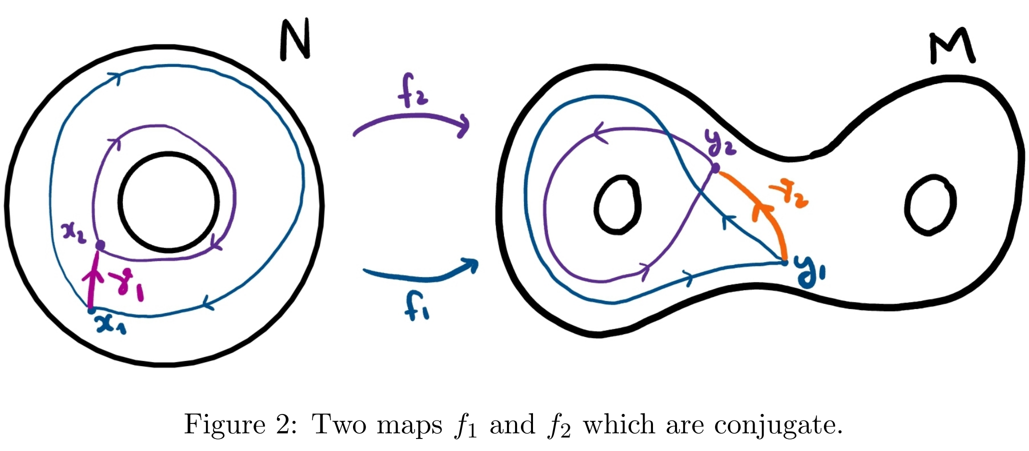

Figure 2: Two maps f_1 and f_2 which are conjugate.

Proposition 4.8. *For f_*^s as in Proposition 4.6, we have*

$$f_*^s \sim_c f_*^t \qquad \forall s,t \in [-1,1] \text{ where this is defined}$$

and thus $[f_*]_c$ is well-defined.

Proof. It suffices to show that the two loops $f \circ \gamma_i^s$ and $f \circ \gamma_i^t$ are homotopic for some arbitrary fixed i.

Write $g = f \circ \psi_i := S^1 \times [-1,1] \to M$, and $\gamma_s = g(\cdot, s) : S^1 \to M$. We want to show that all the absolutely continuous γ_s's are homotopic to each other in M.

Now for any two $s,t \in [-1,1]$, we know that $S^1 \times s$ and $S^2 \times t$ are homotopic to each other in the sense that the following loop is contractible in $S^1 \times [-1,1]$:

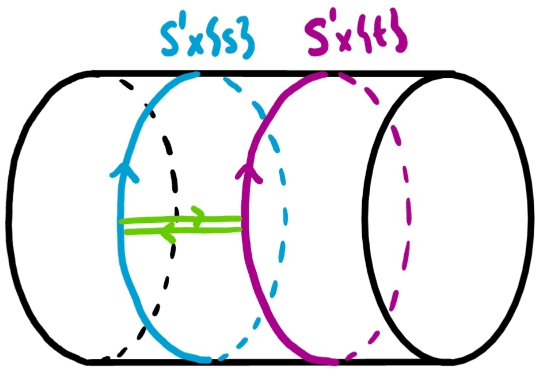

Figure 3: The loop we contract on the cyllinder.

But then we obtain a map $h : D \to M$ given by identifying the inside of the loop with the disc D and composing with g.

Then given any $\varepsilon > 0$, there is some $\delta > 0$ such that whenever $|s-t| < \delta$ we actually must have that the map h has $E(h) < \varepsilon$. By Lemma 4.7, we can deduce the equivalence result for all s,t sufficiently close.

Finally, iterate this result on a partition of the interval $[s,t]$ with step size less than δ to conclude the result for general values of s and t. $\square$

4.2.3 Rigurous definition of the variational problem

With the above definitions in place, we can now properly define the variational problem.

We will define the set

$$\mathcal{F} := \left\{ f \in H^1(N,M) \;\middle|\; [f_*]_c = [\phi_*]_c \right\}$$

and consider the problem of finding

$$\operatorname{argmin}_{\mathcal{F}} E$$

We will denote $I = \inf_{f \in \mathcal{F}} E(f)$. (This is well defined since $\mathcal{F} \neq \emptyset$ and $E(f) \geq 0 \; \forall f$.) Thus. we seek $f \in \mathcal{F}$ with $E(f) = I$.

Theorem 4.9. *There exists $f \in \mathcal{F}$ minimising E i.e. such that $E(f) = I$.*

Proof. Consider a sequence $\{f_j\}$ in $\mathcal{F}$ such that $E(f_j) \searrow I$.

Since M is compact, we must in particular have that the image of the f_j's is bounded in $\mathbb{R}^k$ almost everywhere (uniformly in j). Since N is also compact, we must then have that there is a uniform L^2

bound on the f_j's. But also $E(f_j)$ converges, thus it's bounded, so we obtain a uniform bound on the H^1 norm of the f_j's.

By Remark 4.5, WLOG passing to a subsequence, there exists a map $f \in H^1(N, M)$ such that

$$f_j \to f$$

weakly in $H^1(N, M)$, pointwise almost everywhere, and strongly in $L^2(N, M)$, and with

$$E(f) \leq \lim_{j\to\infty} E(f_j) = I$$

In order to establish that $f \in \mathcal{F}$, it remains to show that

$$[f_*]_c = [\phi_*]_c$$

Fix an arbitrary i.

Since $\{f_j\}$ is uniformly bounded in H^1, the same calculation as in the proof of Proposition 4.6 shows that there is a uniform bound

$$\int_{-1}^{1} \left(\int_{S^1} |\mathrm{d}f_j \circ \psi_i|^2 \mathrm{d}t \right) \mathrm{d}s \leq K \qquad \forall j$$

By Fatou's Lemma, this means that

$$\int_{-1}^{1} \liminf_j \left(\int_{S^1} |\mathrm{d}f_j \circ \psi_i|^2 \mathrm{d}t \right) \mathrm{d}s \leq K < \infty$$

Thus, for almost all $s \in [-1, 1]$, we must have that

$$\liminf_j \left(\int_{\gamma_i^s} |\mathrm{d}f_j|^2 \mathrm{d}t \right) = \liminf_j \left(\int_{S^1} |\mathrm{d}f_j \circ \psi_i|^2 \mathrm{d}t \right) < \infty$$

and so we conclude that there is some $K_s < \infty$ such that

$$\int_{\gamma_i^s} |\mathrm{d}f_j|^2 \mathrm{d}t < K_s$$

holds for infinitely many j's, say a subsequence $\{j_k\}$ (dependent on s).

This then implies that the subsequence f_{j_k} is equicontinous on γ_i^s, and thus along a further subsequence we have uniform convergence. By uniqueness of limits, the limit must be f, and thus $f_{j_{k_l}} \circ \gamma_i^s$ is uniformly close to $f \circ \gamma_i^s$, which must mean that they are homotopic in M (up to conjugation) for l sufficiently large.

Thus,

$$[f_*]_c = [\phi_*]_c$$

and hence $f \in \mathcal{F}$ is a minimiser of E. □

Actually, this result can be improved in terms of regularity, to the following statement:

Theorem 4.10. *There exists a harmonic $f \in C^\infty(N, M)$ such that $[f_*]_c = [\phi_*]_c$ and $E(f) = I$.*

Here, by harmonic we mean that the map f is a critical point of the energy functional (in our case, actually a minimiser).

Proof. Let f be a minimiser of E in $\mathcal{F}$.

Consider an arbitrary small disc $\mathcal{D} \subset N$. Since smoothness is a local property, it suffices to show that f is smooth on $\mathcal{D}$. By [Sim96, Lemma 1, Section 2.10], it suffices to show that $f\mid_{\mathcal{D}}$ is energy-minimising amongst all H^1 maps $g : \mathcal{D} \to M$ which agree with f on an open neighbourhood of the boundary of $\mathcal{D}$.

For any g as above, by the boundary condition, we may glue it with $f\mid_{N\setminus\mathcal{D}}$ to obtain an H^1 map $h : N \to M$. If we show that $h \in \mathcal{F}$, then clearly $E(h) \geq E(f)$, which must imply $E(g) \geq E(f\mid_{\mathcal{D}})$ - as desired.

To establish $h \in \mathcal{F}$, note that whenever we pick a generator γ_i for $\pi_1(N)$, we can WLOG force it to avoid the disc $\mathcal{D}$ (since this disc is contractible). Thus, the definition of $[f_*]_c([\gamma_i])$ and $[h_*]_c([\gamma_i])$ will have to agree, and thus $h \in \mathcal{F}$. $\square$

Hence, given any $\phi : N \to M$, we have shown that there is always a smooth harmonic minimiser of energy amongst all H^1 maps with conjugate action on fundamental groups.

4.3 The minimal hypersurface

Now note that the discussion above required us to fix a Riemannian metric on the surface N we considered, and the minimisation problem was solved with respect to this metric. However, since we are merely looking for *some* minimal hypersurface of genus g, we must instead consider the entire moduli space of metrics on $N = \Sigma_g$.

The cases of $g = 1$ and $g \geq 2$ will be slightly different, but we will try to unify them as much as possible.

For $g \geq 2$, we will consider hyperbolic structures on Σ_g. These are diffeomorphims $\Sigma_g \to N$ with N being a hyperbolic surface i.e. a Riemannian manifold of dimension 2 with constant Gaussian curvature -1. It is clear that, under pulling back, any such hyperbolic structure will induce a hyperbolic metric on Σ_g.

However, if two such hyperbolic structures $h_1 : \Sigma_g \to N_1$ and $h_2 : \Sigma_g \to N_2$ are isometric (in the sense that there exists an isometry $f : N_1 \to N_2$ with $h_2 = f \circ h_1$), then we want them to be considered equivalent. This leads us to the following definition:

Definition 4.6 (Teichmüller space)**.** *The* ***Teichmüller space*** T_g *with base surface* Σ_g *is*

$$T_g = \{\, (N, h) \mid N - \text{ Riemannian manifold with curvature } -1, h : \Sigma_g \to N - \text{ diffeomorphism}\} \Big/ \sim$$

where $\sim$ *identifies two pairs* (N_1, h_1) *and* (N_2, h_2) *if and only if the diffeomorphism* $h_2 \circ h_1^{-1}$ *is an isometry.*

Returning to the hyperbolic metrics on Σ_g defined by the hyperbolic structures, the equivalence relation imposed on T_g corresponds to identifying hyperbolic metrics which are related by the pullback under a self-diffeomorphism of Σ_g.

Discarding the information of the diffeomorphism h, we can define

Definition 4.7 (moduli space)**.** *The* ***moduli space*** R_g *with base surface* Σ_g *is the set of all the equivalence classes of hyperbolic surfaces, i.e. the image of the projection map*

$$\begin{aligned} \pi : T_g &\to R_g \\ (N, h) &\mapsto N \end{aligned}$$

Remark 4.11. *The Teichmüller space is a (complete) metric space with the so-called Teichmüller distance. Furthermore, the projection map $\pi : T_g \to R_g$ acts in such a way that the Teichmüller distance also induces a metric on R_g. For more details, see Farb-Margalit [FM12, Chapters 11 and 12]*

Remark 4.12. *The moduli space R_g actually corresponds to many different structures on Σ_g. Immediately from the definition, we see that it is the set of isometry classes of constant curvature metrics on Σ_g.*

Alternatively, it can also be seen as all the conformal structures on Σ_g i.e. the set of all equivalence classes of Riemannian metrics under pointwise conformal rescaling (as introduced in the previous section).

Again, for more details we direct the reader to Farb-Margalit [FM12, Chapter 12].

The definitions for the case $g = 1$ are similar, but we instead consider flat structures on $\Sigma_1 = T^2$ which have unit area.

Now return to our minimisation problem. Recall from Lemma 4.3 that we have a map $\phi_0 : \Sigma_g \to M$ which is injective on fundamental groups. As long as we have a Riemannian structure on Σ_g, we know from Theorem 4.10 that this map can be minimised. Thus, we will consider for any arbitrary point $p = (N, h) \in T_g$, the smooth harmonic map $f_p : N \to M$ which minimises the energy functional amongst all maps $f : N \to M$ which have the same action on fundamental groups as $\phi := \phi_0 \circ h^{-1} : N \to M$ (up to conjugation).

Since the quotienting on T_g is done by isometries and the energy functional E is clearly invariant under these, we actually get a well-defined map on the Teichmüller space T_g given by

$$\begin{aligned} E : T_g &\to \mathbb{R} \\ p &\mapsto E(f_p) \end{aligned}$$

This should be interpreted as "the minimum energy of a map from a given hyperbolic structure to M which preserves the action on fundamental groups". Thus, it makes sense that we want to find a point which achieves the minimum of this energy functional.

We will now cite two well-known results which are necessary to our discussion, but which we will not prove. The first is a result of Sacks and Uhlenbeck [SU81, Theorem 1.8].

Lemma 4.13. *If $p \in T_g$ is a minimum of $E : T_g \to \mathbb{R}$, then the corresponding harmonic map $f_p : N \to M$ is a branched minimal immersion.*

Secondly, from Gulliver [Gul73, Theorem 8.2], we have

Lemma 4.14. *If $f : N \to M$ is a branched minimal immersion and M is a three-dimensional Riemannian manifold, then f is a true (minimal) immersion.*

Thus, finding a minimal immersion $\Sigma_g \to M$ reduces to the following statement:

Proposition 4.15. *The energy functional $E : T_g \to \mathbb{R}$ achieves its minimum at some $p \in T_g$.*

We will now prove this result. Our proof will rely on the following lemma:

Lemma 4.16. *Given any $K > 0$, the set*

$$S = \{\, \pi(p) \mid p \in T_g, E(p) \leq K \,\} \subset R_g$$

is compact, and thus also sequentially compact.

Since the proof of this lemma is rather technical and long, we will defer it to the end of this section. For now, we will assume it is true and use it to prove Proposition 4.15.

Proof of Proposition 4.15. Denote $I = \inf_{p \in T_g} E(p)$ and consider a sequence $\{p_i\}$ in T_g with $E(p_i) \searrow I$. Pick representatives (N_i, h_i) for each p_i. We will construct a convergent subsequence of these points, and the limit will be the desired minimiser.

Clearly there exists some $K > 0$ with $E(p_i) \leq K$ for all i. By Lemma 4.16 we can pass to a subsequence and assume that N_i converges to some $N \in R_g$. Identify now each N_i with the conformal structure σ_i on Σ_g associated with it, and consider instead the minimisers f_{p_i} as functions $\Sigma_g \to M$.

In fact, since the conformal structure converges and the continuity estimates for each f_{p_i} depend continuously on the metric we put on Σ_g, this family of maps is equicontinuous. Thus, we can use the Arzelà-Ascoli Theorem to pass to a subsequence

$$f_{p_i} \to (f : \Sigma_g \to M)$$

for f harmonic with respect to the limit conformal structure σ, and such that

$$E(f) \leq \lim_{i \to \infty} E(f_{p_i}) = I$$

Thus $E(f) = I$, and the conformal structure σ gives the desired point in T_g which minimises the energy functional. □

So now we know we have a minimal surface $f : \Sigma_g \to M$ given by the map f_p at the minimum $p \in T_g$ of $E : T_g \to \mathbb{R}$. Finally, we wish to establish that this map is actually a *stable* minimal immersion i.e. that it minimises area.

Proposition 4.17. *The map $f : \Sigma_g \to M$ obtained above is a stable minimal immersion.*

Proof. Consider an arbitrary immersion $f' : \Sigma_g \to M$. Let σ' be the conformal structure on Σ_g obtained by pulling back the metric on M by f', and let $p' \in T_g$ be the corresponding point in Teichmüller space. Then clearly

$$E(p') \leq E(f')$$

We want to show that actually

$$\mathrm{Area}(f') = \frac{1}{2} E(f')$$

for all such f'. If this holds, then any sufficiently small perturbation $f' : \Sigma_g \to M$ of the immersion f is also an immersion. Thus, we have

$$\mathrm{Area}(f') = \frac{1}{2} E(f') \geq \frac{1}{2} E(p') \geq \frac{1}{2} E(p) = \frac{1}{2} E(f) = \mathrm{Area}(f)$$

meaning f defines a stable minimal immersion.

Now, since $f' : (\Sigma_g, (f')^* g) \to (M, g)$ is an isometry, an easy linear algebra argument shows that actually

$$\langle \mathrm{d}f'_x, \mathrm{d}f'_x \rangle = \dim(\Sigma_g) = 2$$

for any $x \in \Sigma_g$.

This then integrates to imply precisely that

$$E(f') = \int_{\Sigma_g} \langle \mathrm{d}f', \mathrm{d}f' \rangle^2 \, \mathrm{d}v = 2 \int_{\Sigma_g} \mathrm{d}v = 2 \, \mathrm{Area}(f')$$

as required. □

Finally, we will go back and prove Lemma 4.16.

Proof of Lemma 4.16. Firstly, we note that, since M is compact, its injectivity radius is positive i.e. there exists some $\varepsilon > 0$ such that the exponential map is well-defined on balls of radius ε around any point in M. An immediate implication of this is that any loop in M which has length smaller than some $\delta = \delta(\varepsilon) > 0$ must be contained in the image of the exponential map at a point, and thus it must be contractible in M.

The idea of this proof will be to integrate this bound on a one-parameter family of non-contractible loops, and manipulate the integrals to relate them to the energy of the map f_p, which we have a bound on.

We will have to treat the cases $g = 1$ and $g \geq 2$ separately, due to some results we want to appeal to only working for Riemann surfaces uniformised by the upper half-plane.

- For $g = 1$, we have explicit formulations for T_g and R_g. We know that T_g is the space of all complex tori, which are of the form $\mathbb{C}/\Lambda$ for some lattice $\Lambda \subset \mathbb{C}$.

  Upon applying rotations and reflections of $\mathbb{C}$, the lattice Λ can be written as $\Lambda = \mathbb{Z} \oplus w\mathbb{Z}$ for some $w \in \mathcal{H} = \{ z \in \mathbb{C} \mid \operatorname{Im}(z) > 0 \}$, and thus T_g can be identified with the upper half-plane $\mathcal{H}$.

  The group $SL(2, \mathbb{Z})$ acts on $\mathcal{H}$ via $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot z = \frac{az+b}{cz+d}$, and we recall that the moduli space R_g is given precisely by the quotient
  $$R_g = \mathcal{H}/SL(2, \mathbb{Z})$$
  under this action.

  We can thus identify it with the fundamental domain
  $$\mathcal{D} = \left\{ z \in \mathcal{H} \;\middle|\; -\frac{1}{2} < \operatorname{Re}(z) \leq \frac{1}{2}, |z| \geq 1, \text{ and } \operatorname{Re}(z) \geq 0 \text{ if } |z| = 1 \right\}$$

  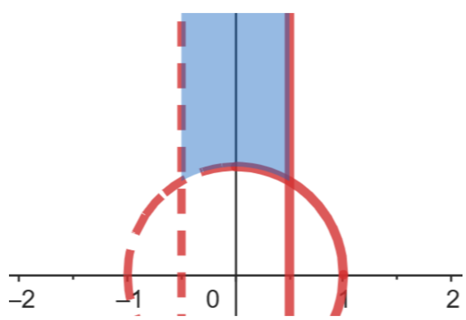

  Figure 4: Fundamental domain of R_1

  We will claim that, picking representatives in this fundamental domain, having $E(p) \leq K$ requires
  $$\operatorname{Im}(z) \leq Y$$
  for some Y sufficiently large. If this holds, then we get that
  $$S \subset \pi(\{\operatorname{Im}(z) \leq Y\} \cap \overline{\mathcal{D}})$$
  which is compact (as continuous image of a compact set).

Consider now some $q \in S$ and pick a representative $\tau \in \mathcal{D}$ corresponding to q. We want to show that $\mathrm{Im}(\tau)$ is bounded above.

Fix a height $y \in [0, \mathrm{Im}(\tau)]$ and consider the loop γ_y given by the horizontal line at height y in the fundamental parallelogram of the torus i.e. $\gamma_y(x) = (x, y)$ for $x \in [y\theta, y\theta + 1]$, where $\theta = \frac{\mathrm{Re}(\tau)}{\mathrm{Im}(\tau)}$.

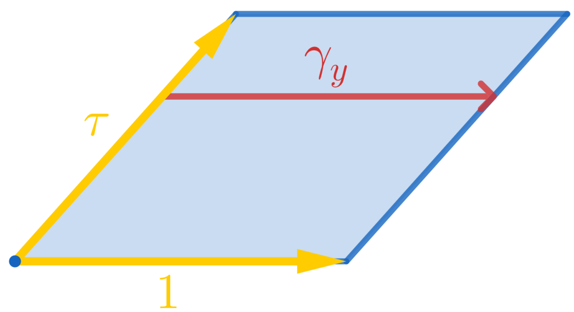

Figure 5: Loop γ_y

Since this is not contractible in the complex torus, its image under f is also not contractible in M. Hence, we have a bound for its length:

$$\begin{aligned}
\delta &\leq \ell(\gamma_y) \\
&= \int_{y\theta}^{y\theta+1} \left\| \mathrm{d}(f_p) \circ \gamma_y'(x) \right\| \mathrm{d}x \\
&\leq \int_{y\theta}^{y\theta+1} |\mathrm{d}(f_p)| \, \mathrm{d}x \qquad \text{since } \left|\gamma_y'(x)\right| = 1 \\
&\leq \left(\int_{y\theta}^{y\theta+1} |\mathrm{d}(f_p)|^2 \, \mathrm{d}x \right)^{\frac{1}{2}} \qquad \text{by Cauchy-Schwarz}
\end{aligned}$$

Squaring and integrating this over all values of y, we must have

$$\delta^2 \, \mathrm{Im}(\tau) \leq \int_0^{\mathrm{Im}(\tau)} \int_{y\theta}^{y\theta+1} |\mathrm{d}(f_p)|^2 \, \mathrm{d}x \, \mathrm{d}y \leq E(p) = K$$

and thus we obtain a bound

$$\mathrm{Im}(\tau) \leq \frac{K}{\delta^2}$$

which is exactly of the form we wanted.

- For $g \geq 2$, we will use the following result from Mumford [Mum71, Corollary 3]:

Lemma 4.18. *For any given $\varepsilon > 0$, the set*

$$\{\, q \in R_g \mid \textit{in the Poincaré metric, all closed geodesics on } X \textit{ have length } \geq \varepsilon \,\}$$

is a compact subset of R_g.

Thus, it remains to show that being an element of S implies that the length of closed geodesics is bounded below. (Note: clearly S itself is closed, and so it suffices to show that it's a subset of a compact set.).

This argument has a similar flavour to the one we used for $g = 1$.

Fix a point $p = N \in R_g$ and consider a closed geodesic γ on N of length ℓ. By the main theorem of Keen [Kee74], we can find a *collar* of γ i.e. an isometric copy of the following region in the upper half-plane

$$\mathcal{C} = \left\{ z \in \mathbb{C} \;\middle|\; 1 < |z| < e^{\ell}, \theta < \arg(z) < \pi - \theta \right\}$$

with the two curved edges glued along points with the same argument. This identification is done in such a way that the loop iy in the collar is mapped to γ.

Here, θ is an angle such that the total area $2\ell\cot(\theta)$ of the collar is equal to $\frac{8}{\sqrt{5}}$. More precisely, this means $\theta = \tan^{-1}\left(\frac{\ell\sqrt{5}}{4}\right)$

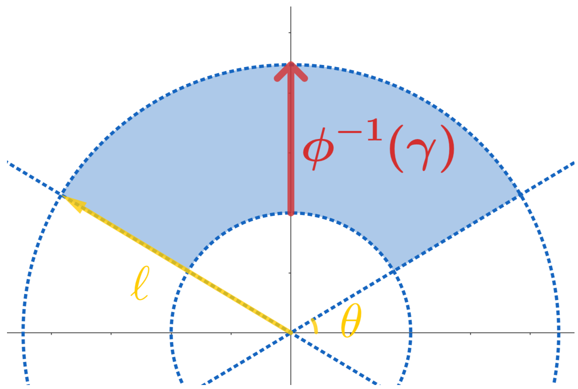

Figure 6: Collar in the upper half-plane

Let $\phi : \mathcal{C} \to N$ be this isometric embedding.

For an arbitrary $\alpha \in (\theta, \pi - \theta)$, we can consider the loop γ_α given by $\gamma_\alpha(t) = te^{i\alpha}$ for $t \in [1, e^{\ell}]$. Since these are all homotopic to γ in the collar and γ is not contractible in M, the image of each γ_α under f_p cannot be contractible in M either. Thus, we have a lower bound on their length:

$$\begin{aligned}
\delta^2 &\leq \left(\int_1^{e^l} \|\mathrm{d}(f_p) \circ \dot{\gamma}_\alpha(t)\| \,\mathrm{d}t \right)^2 \\
&\leq \left(\int_1^{e^l} |\mathrm{d}(f_p)| \frac{1}{r \sin\alpha} \,\mathrm{d}r \right)^2 && \text{since } \|\dot{\gamma}_\alpha(t)\|_{\text{hyp}} = \frac{1}{r\sin\alpha} \\
&\leq \int_1^{e^l} \frac{1}{r} \,\mathrm{d}r \cdot \int_1^{e^l} |\mathrm{d}(f_p)|^2 \frac{1}{r \sin^2\alpha} \,\mathrm{d}r && \text{by Cauchy-Schwarz} \\
&\leq \ell \int_1^{e^l} |\mathrm{d}(f_p)|^2 \frac{1}{r \sin^2\alpha} \,\mathrm{d}r
\end{aligned}$$

and so integrating this over all $\alpha \in (\theta, \pi - \theta)$ we obtain

$$\frac{\delta^2(\pi - 2\theta)}{\ell} \leq \int_{\mathcal{C}} |\mathrm{d}(f_p)|^2 \frac{1}{r\sin^2\alpha} \,\mathrm{d}r\,\mathrm{d}\theta \leq E(f_p) = K$$

However, since $\theta = \theta(\ell)$, this does not immediately rearrange to give us a bound as in the case $g = 1$. Instead, we note that the above tells us

$$\frac{\pi - 2\tan^{-1}\left(\frac{\ell\sqrt{(5)}}{4}\right)}{\ell} \leq \frac{K}{\delta^2}$$

and a check of the derivatives tells us that the left hand side is strictly decreasing and goes to $+\infty$ as $\ell \to 0$. Thus, the upper bound on the LHS function in fact gives us a lower bound on ℓ, which is what we wanted.

$\square$

4.4 No minimal hypersurface of positive genus

Finally, to conclude our proof of Theorem 4.1, we need the following result:

Theorem 4.19. *Let M be a compact oriented 3-manifold with positive scalar curvature.*

Then M has no closed immersed stable minimal two-dimensional orientable submanifolds of positive genus.

Here, by submanifold we mean an immersed submanifold. We will also refer to submanifolds of codimension 1 as hypersurfaces.

Since we have already shown how $\pi_1(M)$ having a subgroup isomorphic to $\pi_1(\Sigma_g)$ leads to the existence of a minimal hypersurface of positive genus, we conclude that this is not compatible with having PSC, and thus Theorem 4.1 holds.

Proof of Theorem 4.19. Suppose N is a hypersurface as in the statement of the theorem. We will show that this leads to a contradiction.

Let $R_{ij;kl}$ be the curvature tensor of M, $\widetilde{R}_{ij;kl}$ be that of N, and Π_{ij} be the second fundamental form of N. Similarly, denote by R and $\widetilde{R}$ the scalar curvatures of M and N respectively.

Let e_n be the unit normal vector of N and $\phi \in C^\infty(N)$ be an arbitrary smooth function. Consider the deformation given by ϕe_n. We then have the following result:

Lemma 4.20. *If N is a stable orientable minimal hypersurface in M and $\phi \in C^\infty(N)$ is an arbitrary smooth function, then the deformation under ϕ must satisfy*

$$-\int_N \left(\mathrm{Ric}\,(e_n, e_n) + \sum_{i,j} \Pi_{ij}^2\right) \phi^2 + \int_N |\nabla \phi|^2 \geq 0 \tag{3}$$

where $\mathrm{Ric}\,(e_n, e_n)$ is the Ricci curvature of M in the direction of e_n.

On the other hand, a corollary of the Gauss-Codazzi equations is that

$$R = \widetilde{R} + 2\mathrm{Ric}\,(e_n, e_n) + \sum_{i,j=1}^{n} \Pi_{i,j}^2 - H^2$$

where H is the mean curvature of N. Deducing this from the standard form of the Gauss-Codazzi equations is a straightforward calculation which we will omit, but which can be found for example on Wikipedia [24].

Since N is minimal, we know that $H = 0$, and so we get that

$$\mathrm{Ric}\,(e_n, e_n) + \sum_{i,j} \Pi_{ij}^2 = \frac{R}{2} - \frac{\widetilde{R}}{2} + \frac{1}{2} \sum_{i,j} \Pi_{ij}^2$$

We substitute this in (3) to obtain

$$\int_N \frac{R\phi^2}{2} - \int_N \frac{\widetilde{R}\phi^2}{2} + \frac{1}{2} \int_N \phi^2 \sum_{i,j} \Pi_{ij}^2 \leq \int_N |\nabla \phi|^2$$

Since M has PSC, we know that $R > 0$. Furthermore, $\sum_{i,j} \Pi_{ij}^2 \geq 0$, and thus

$$-\int_N \frac{\widetilde{R}\phi^2}{2} < \int_N |\nabla\phi|^2 \tag{4}$$

for all ϕ smooth, not identically 0.

In particular, picking $\phi \equiv 1$, we must have

$$-\int_N \frac{\widetilde{R}}{2} < 0$$

On the other hand, by Gauss-Bonnet we know

$$\int_N \mathcal{K} = 2\pi\chi(N) \leq 0 \quad \text{(since } N \text{ has positive genus)}$$

where $\mathcal{K}$ is the Gauss curvature of N.

Finally, recall that Gauss and scalar curvature of 2-manifolds relate by $\mathcal{K} = \frac{\widetilde{R}}{2}$, and so the two inequalities above contradict each other. □

Proof of Lemma 4.20 (sketch). Consider the one-parameter family of hypersurfaces given by "$\Sigma_t = \Sigma + t\phi e_n$" for $t \in (-\varepsilon, \varepsilon)$. Denote by A_t the area of Σ_t.

Since $\Sigma = \Sigma_0$ is a stable minimal surface, we must have that $t = 0$ is a local minimum for A_t, so

$$\left.\frac{\mathrm{d}^2 A_t}{\mathrm{d}t^2}\right|_{t=0} \geq 0$$

A somewhat long calculation, which can be found in [CM99, Chapter 1, Section 8] allows us to express this second derivative as

$$\left.\frac{\mathrm{d}^2 A_t}{\mathrm{d}t^2}\right|_{t=0} = -\int_\Sigma \phi \left(\Delta\phi + \sum_{i,j} \Pi_{ij}^2 \phi + \mathrm{Ric}\left(e_n, e_n\right)\phi\right)$$

Using integration by parts, we can rewrite the first term as

$$-\int_\Sigma \phi\Delta\phi = \int_\Sigma |\nabla\phi|^2$$

and thus obtain the desired result. □

5 Positive Scalar Curvature in higher dimensions

We have so far seen in Theorem 4.1 that given any compact connected orientable PSC manifold M of dimension 3, its fundamental group $\pi_1(M)$ must not have any subgroup isomorphic to $\pi_1(\Sigma_g)$ for $g \geq 1$.

We now want to generalise this result to a topological restriction in higher dimensions. This section is inspired by the first part of Schoen-Yau's work [SY79b].

5.1 Minimal submanifolds of positive scalar curvature

The key result which will allow us to do this is the following:

Theorem 5.1. *Consider a compact oriented manifold (M, g) of dimension $n > 3$ which has PSC. If Σ is a closed stable orientable minimal hypersurface, then Σ is PSC.*

Remark 5.2. *Here, we do not claim that Σ has PSC under the metric inherited from M, but only that it does admit* some *metric with PSC.*

Proof. Using the exact same argument as in the proof of Theorem 4.19, we can show that for any Σ as above, we must have that

$$-\int_\Sigma \frac{\widetilde{R}\phi^2}{2} < \int_\Sigma |\nabla\phi|^2 \tag{5}$$

for all Φ smooth, not identically 0 (where $\widetilde{R}$ is the scalar curvature of Σ as a submanifold of M).

We will now want to rescale the induced metric on Σ to one with positive scalar curvature. We have seen in Lemma 3.14 that it suffices to show that the first eigenvalue of the differential operator

$$L = -\Delta + \frac{n-3}{4(n-2)}\widetilde{R}$$

is positive.

Let λ_1 be this first eigenvalue, and ϕ be a positive eigenfunction corresponding to λ_1 (which exists by Lemma 3.5).

Now, since $L\phi = \lambda_1\phi$, we must have:

$$-\frac{1}{2}\widetilde{R}\phi^2 = -\frac{2(n-2)}{n-3}\phi\Delta\phi - \frac{2(n-2)}{n-3}\lambda_1\phi^2$$

Substituting this into (5) and using integration by parts on the $\phi\Delta\phi$ term, we get

$$\frac{2(n-2)}{n-3}\int_\Sigma |\nabla\phi|^2 - \frac{2(n-2)}{n-3}\lambda_1\int_\Sigma \phi^2 < \int_\Sigma \left|\nabla\phi^2\right|$$

which rearranges to

$$0 \leq \frac{n-1}{2(n-2)}\int_\Sigma |\nabla\phi|^2 < \lambda_1\int_\Sigma \phi^2$$

Since $\phi > 0$, we must have that $\lambda_1 > 0$, and so we're done. □

5.2 Application to proving the Geroch conjecture

Finally, we will give a proof of the Geroch conjecture in dimensions $3 \leq n \leq 7$:

Corollary 5.3 (Geroch Conjecture)**.** *The Geroch conjecture holds for all $3 \leq n \leq 7$ i.e. T^n is not PSC for all $3 \leq n \leq 7$.*

For this, we will need the following standard result in geometric measure theory, found for example in a set of notes by Chodosh [Cho, Theorem 4.2].

Proposition 5.4. *Suppose (M, g) is a closed, oriented n-dimensional manifold and $2 \leq n \leq 7$. Then for any $\alpha \in H_{n-1}(M, \mathbb{Z})$, we can find codimension one compact (connected) stable minimal submanifolds $\Sigma_1, \ldots, \Sigma_k$ such that*

$$\alpha = [\Sigma_1] + \cdots + [\Sigma_k] \in H_{n-1}(M, \mathbb{Z})$$

Now, Theorem 5.1 tells us that (assuming M has PSC) each Σ_i is PSC. But then this will restrict the topology of each Σ_i, in turn allowing us to inductively restrict the possible topology of M.

Proof of Corollary 5.3 - Geroch Conjecture. We will prove this by induction on n. The case $n = 3$ was already shown in Corollary 4.2.

Suppose it holds for some $3 \leq n \leq 6$, but that T^{n+1} is PSC. Consider the smooth inclusion $\iota : T^n \hookrightarrow T^{n+1}$ given by $(\theta_1, \ldots, \theta_n) \mapsto (\theta_1, \ldots, \theta_n, 0)$ (where θ_i are the coordinates on each S^1).

Then taking $\alpha = [T^n] \in H_n(T^n; \mathbb{Z})$, we can see that $\iota_* \alpha \neq 0 \in H_n(T^{n+1}; \mathbb{Z})$.

Minimising in this homology class as in Proposition 5.4, we find a minimal T^n embedded in T^{n+1}. By Theorem 5.1, this minimal T^n must be PSC, which contradicts the induction hypothesis. □

In fact, the same proof allows us to establish the following more general statement (see [SY79b, Corollary 1]):

Corollary 5.5. *Let M_n be an n-dimensional compact orientable manifold, with $3 \leq n \leq 7$. Suppose that for every $3 \leq i < n$ there exists a manifold M_i and a map $M_{i+1} \to M_i$ which pulls back the fundamental class of M_i to a non-zero element in $H_i(M_{i+1}; \mathbb{Z})$, and that M_3 is not PSC.*

Then M_n is not PSC.

6 Conclusion

We will conclude the essay by returning to the question we posed in the introduction:

Question. *Given a compact orientable manifold M and a smooth function $f \in C^\infty(M)$, is there a Riemannian metric g on M such that the scalar curvature of (M, g) is equal to f?*

We have seen that conformal rescalings allow us to use a positive answer for some f_0 to answer to question affirmatively for a wide range of other function f. In particular, from Elíasson's result [Elí71] that metrics with negative total scalar curvature exist we were able to conclude that the answer to the above question is yes whenever f is not everywhere non-negative.

Setting aside the case of manifolds which admit metrics with scalar curvature identically zero, we have then claimed (and proved one direction of the implication) that the problem reduced to whether or not a given manifold M is PSC.

We then investigated this question in the case of 3-dimensional manifolds in Section 4, where we found that the fundamental group of any PSC manifold cannot contain a subgroup isomorphic to the fundamental group of a genus g surface. In Section 5, we constructed an inductive argument for dimensions $4 \leq n \leq 7$, by claiming that stable minimal hypersurfaces of PSC manifolds must also be PSC.

In particular, our inductive argument applied to the Geroch conjecture, proving that T^n is not PSC for all $3 \leq n \leq 7$. On the other hand, the existence of flat metrics on T^n is a well-known result, and thus we conclude that T^n must lie in the second case of our trichotomy in Theorem 3.1 i.e.

$$\begin{gathered} f \in C^\infty(M) \text{ is the scalar curvature of a Riemannian metric on } M \\ \iff f \equiv 0 \text{ or } f \text{ not everywhere} \geq 0 \end{gathered}$$

When it comes to finding manifolds in the third case, namely those which are PSC, and thus admit *any* scalar curvature, we have a very simple example given by S^n, where the standard metric coming from its embedding into Euclidean $\mathbb{R}^{n+1}$ has constant positive scalar curvature. Furthermore, in the latter half of [SY79b], Schoen and Yau give ways in which one can do surgery on a PSC manifolds to obtain new PSC manifolds, thus providing tools for constructing more manifolds in this class. Similar results were also obtained independently by Gromov and Lawson in [GL80b].

However, one main drawback of the argument we have presented in Section 5 is that we required the regularity of hypersurfaces, which thus restricted us to only considering manifolds of dimension $n \leq 7$. There have been other approaches the question posed, which have avoided the dimension restriction arising from the regularity of minimal surfaces, but which have instead had to restrict to a special class of manifolds. Two such examples are:

- In the case of compact *spin* manifolds, the existence of metrics with PSC implies that the $\widehat{A}$-genus of the manifold must be zero, as shown by Lichnerowicz [Lic63]. This is done by first showing that compact PSC spin manifolds have no harmonic spinor, and then applying the Atiyah–Singer index theorem to the Dirac operator. Later, Gromov and Lawson [GL80a] generalised this condition to what they called enlargeable manifolds. One such manifold is precisely T^n, so their work can be used to answer the Geroch conjecture for dimensions $n > 7$ as well.

- More recently, Chodosh and Li [CL24] proved that closed manifolds of dimensions 4 or 5 which are *aspherical* (i.e. have a contractible universal cover) are never PSC.

Finally, for 2-dimensional manifolds, necessary conditions on f in terms of the Euler characteristic of M are easy to deduce from the Gauss-Bonnet theorem, in the same way we did in Corollary 2.2. Furthermore, Kazdan and Warner proved in [KW74] that these conditions are also sufficient when dealing with $\chi(M) \leq 0$.

So, while a complete classification of what manifolds admit metrics with positive scalar curvature has not yet been found, there are many different results highlighting necessary topological conditions that PSC manifolds must obey, as well as surgery results useful in the construction of examples with positive scalar curvature.

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